TRANSCENDENCE OF e.

Theorem. (Hermite 1873). *The number* **e**, *the base of natural logarithms, is transcendental.*

Proof. (1) Use integration by parts on the integral $\int_0^x e^{-t}f(t)dt$ and multiply the resulting equality by e^x to obtain:

$$e^x \int_0^x e^{-t}f(t)dt = e^x f(0) - f(x) + e^x \int_0^x e^{-t}f'(t)dt.$$

Note that the integral on the right-hand side is obtained from the one on the left-hand side by replacing f by f'. This means that if we write the same identity for successive derivatives f', f'', f''', etc. in place of f and add the results, there will be cancellations. In particular, assume now f is a polynomial. Define:

$$F(x) = \sum_{j=0}^{\infty} f^{(j)}(x),$$

which is a finite sum for polynomial f. Then (after cancellation) the result of adding the identities above for the functions f, f', f'', etc. can be written using F in the form:

$$e^x \int_0^x e^{-t}f(t)dt = e^x F(0) - F(x). \quad (*)$$

What are we using about e here? The fact that the function e^x has the value 1 at $x = 0$ and is equal to its own derivative. (It is the only differentiable function with this property.)

(2) Now assume (by contradiction) e is algebraic. By definition, this means there is a polynomial $p(t)$ with integer coefficients a_j ($a_0 \neq 0$) and degree $n \geq 1$ so that $p(e) = 0$:

$$a_0 + a_1 e + a_2 e^2 + \ldots + a_n e^n = 0. \quad (1)$$

(Note we allow $n = 1$, so in particular we are not assuming *a priori* that e is irrational, a fact which had been proved by Liouville about 20 years earlier.)

Write the identity (*) for $x = k = 0, 1, 2, \ldots, n$, multiply by a_k and add the results. We obtain:

$$\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t}f(t)dt = F(0)\sum_{k=0}^{n} a_k e^k - \sum_{k=0}^{n} a_k F(k).$$

Using the equation (1), we may re-write this in the form:

$$\sum_{k=0}^{n} a_k F(k) = -\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)dt. \quad (**)$$

(3) Note that, so far, we are still free to choose the polynomial $f(t)$. The *idea* of the proof now is to choose f so that the left-hand side of (**) is a non-zero integer (hence greater than or equal to one in absolute value), while the right-hand side is small, giving a contradiction. Hermite's inspired choice for f is:

$$f(t) = \frac{1}{(p-1)!} t^{p-1} g(t)^p, \text{ where } g(t) = (t-1)(t-2)\dots(t-n) \quad (2)$$

and p is a prime number that we're free to choose as large as needed. Denote by A the maximum absolute value of $tg(t)$, over the interval $[0, n]$. Then we have for the right-hand side of (**) the bound:

$$|\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)dt| \leq \frac{1}{(p-1)!} (\sum_{k=0}^{n} |a_k|) e^n n A^p,$$

(where $A^p/(p-1)!$ is an upper bound for the integrand over the interval of integration $[0, n]$). On the right-hand side of this estimate, the coefficients a_k, the degree n and the number A are fixed, independent of p. Since the factorial function grows faster than any exponential, this implies the right-hand side of (**) can be made as small as desired (say, less than 1/2), by choosing the prime p large enough.

(4) To conclude the proof, we just need to check that $\sum_{k=0}^{n} a_k F(k)$ is a nonzero integer. Here $F(k)$ is the sum of the values of f and all its derivatives at an integer k, and the polynomial g has integer coefficients, so the denominator $(p-1)!$ in f is our potential problem. We first check what happens at $k = 0$. Since $t = 0$ is a zero of $f(t)$ of multiplicity $p - 1$, the Taylor expansion of f at $t = 0$ (which is really a finite sum, since f is a polynomial) has the form:

$$f(t) = \frac{1}{(p-1)!} f^{(p-1)}(0) t^{p-1} + \frac{1}{p!} t^p f^{(p)}(0) t^p + \ldots + \frac{1}{j!} f^{(j)}(0) t^j + \cdots \quad (3)$$

Clearly $f^{(j)}(0) = 0$ for $j < p - 1$, and comparing coefficients of t^{p-1} in (2) and (3) we see that $f^{(p-1)}(0)$ is the p^{th} power of the constant term in the polynomial $g(t)$:

$$f^{(p-1)}(0) = [(-1)^n n!]^p.$$

So $f^{(p-1)}(0)$ is an integer. At this point we impose the last largeness requirements on p: $p > n$ and $p > |a_0|$. (The reason for the second requirement will be seen below.) Since p is a prime number greater than n, it does not occur in the prime factorization of $n!$. Hence we know that the integer $f^{(p-1)}(0)$ is not a multiple of p. As for the higher-order derivatives of f at zero, again comparing coefficients of powers of t in (2) and (3) we see that, for $j \geq p$:

$$f^{(j)}(0) = \frac{j!}{(p-1)!} \times (\text{ coefficient of } t^{j-(p-1)} \text{ in } g(t)^p),$$

and the quantity in parenthesis is certainly an integer. Since $j \geq p$, the number $j!/(p-1)!$ is also an integer, and a multiple of p. This shows $F(0)$ is a non-zero integer (since it is the sum of integers, only one of which is not a multiple of p), and likewise for $a_0 F(0)$.

(5)The proof that $F(k)$ is an integer for $k = 1, 2, \ldots n$ is similar, but considering Taylor expansions at k. Take, for example, $k = 2$ (if $n \geq 2$). Since 2 is a zero of f with multiplicity p, certainly $f^{(j)}(2) = 0$ for $j < p$, and we have:

$$f(t) = \frac{1}{p!} f^{(p)}(2)(t-2)^p + \ldots + \frac{1}{j!} f^{(j)}(2)(t-2)^j + \ldots (4).$$

Comparing coefficients in (2) and (4), we find, for $j \geq p$:

$$f^{(j)}(2) = \frac{j!}{(p-1)!} \times \text{ coefficient of } (t-2)^{j-p} \text{ in } t^{p-1} g(t)^p.$$

(By a change of variable, we can always express the polynomial $t^{p-1}g(t)^p$ in terms of powers of $(t-2)$, instead of powers of t.) Note that, for $j \geq p$, this clearly shows $f^{(j)}(2)$ is an integer, and a multiple of p. So $F(2)$ is also an integer, and multiple of p; and likewise for $k = 1, 3, \ldots n$, showing that $\sum_{k=1}^{n} a_k F(k)$ is an integer, and multiple of p; and recalling the result of part (4) we even have that the sum starting at zero, $S = \sum_{k=0}^{n} a_k F(k)$, is an integer. Can it be zero? Well, the only one of its terms that is *potentially* not a multiple of p is $a_0 F(0)$. We established in (4) that $F(0)$ is not a multiple of p, and since p is a prime, if S were zero this would force a_0 to be a multiple of p. Aaahhh...*that's* why we imposed the condition $p > |a_0|$- to make this impossible. This concludes the proof!

(Proof based on the book by A.O. Gelfond, 'Transcendental and algebraic numbers' (Dover 1960), pp. 42-44, with cosmetic changes by A.F.)

EXERCISES (due Friday, 7/7)

1. Show, using Hermite's theorem, that the natural logarithm of a positive integer (different from 1) cannot be a rational number.

2. Recall that an irrational number is said to be *constructible* (with ruler and compass) if it can obtained from positive integers by iterated application of field operations and taking square roots. Explain why constructible numbers are algebraic, and give an example of an algebraic number that is not constructible.

The following problems refer to chapter 9 in [Dunham] (Weierstrass)

3. Explain, as precisely as possible and based only on the definition of uniform convergence and/or Fig 9.6, why the sequences of functions in Fig 9.2/3 and Fig 9.4/5 do not converge uniformly.

4. The Weierstrass example shows that, even if a sequence f_n of differentiable functions converges uniformly to f, it does not follow that the derivatives of f_n converge to the derivative of f. But there are simple ways to give such an example.

(i)Show that the sequence of functions $f_n(x) = x^{2n}(1 - x^{2n})$ does not converge uniformly in $[0, 1]$ (*Hint:* find the maximum of f_n in $[0, 1]$.)

(ii) Show that the sequence $g_n(x) = \frac{x^{2n+1}}{2n+1} - \frac{x^{4n+1}}{4n+1}$ converges uniformly to $g \equiv 0$ in $[0, 1]$, but that its derivatives converge to a discontinuous, non constant function. Thus, in this case, $\lim g_n' \neq (\lim g_n)'$, even though g_n converges uniformly.